ABSTRACT

This thesis studies the theory of fuzzy topological spaces and its relation to general topological spaces. We begin with an introduction to fuzzy sets and then work on two well-known definitions for fuzzy topology: Chang's original definition and Lowen's improved version, which solves key problems in terms of continuity and compactness. Furthermore, we study the mutually generative relations between fuzzy and general topologies by the function ω and ι, emphasizing their connections to each other. Finally, we systematically analyze the concepts of continuity and compactness under these two definitions and study in depth the relations and implications between them.

Keywords: Fuzzy Set, Fuzzy Topological Space, Fuzzy Continuous, Fuzzy Compactness, Topological Space.

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LIST OF SYMBOLS / ABBREVIATIONS

μ	:	Membership Function
χ	:	Characteristic Function
$\emptyset$	:	Empty Set
$\big(X, \mu(x)\big)$	:	Fuzzy Set
$\big(X, \chi(x)\big)$	:	Crisp Set
$\emptyset_X$	:	Empty Fuzzy Set
$\sqcup_X$	:	Fully Included Fuzzy Set
$(X, \mathscr{T})$	:	Topological Space
(X, δ) or fts	:	Fuzzy Topological Space
$nbhd$	:	Neighbourhood of A Fuzzy Set
$\mathscr{J}(X)$	:	Set of All Topologies on X
$\mathscr{W}(X)$	:	Set of All Fuzzy Topologies on X
f	:	Function Between Two Fuzzy Sets
$\tilde{f}$	:	Function Between Two Fuzzy Topological Spaces
F-continuous	:	Fuzzy Continuous
F-compact	:	Fuzzy Compact

CHAPTER 1

INTRODUCTION

The theory of fuzzy topological spaces is an extension of general topology. Generally, topological concepts use exact binary logic to express whether they belong to a set or not. However, many real-life concepts are not 'either/or'; for example, the boundaries of the ocean change with the ebb and flow of the tides, and therefore cannot simply be represented by a line. Pests and diseases tend to start from a single original spot and gradually spread to the surrounding areas, and the degree of damage is also gradually decreasing, so a disaster cannot simply be evaluated by whether it is affected or not.

To express information more objectively, there is an acute need to solve the problem of applications involving imprecise or uncertain data. This need for a more detailed and nuanced understanding of the membership of sets led to the development of fuzzy set theory, introduced by Lotfi Zadeh in the 1960s [Zad65], which in turn led to the study of fuzzy topology.

The theory of fuzzy topological spaces originated from the fundamental work of Zadeh, and mathematicians such as C. L. Chang and Robert Lowen played a key role in establishing and refining the field. In 1968, Chang introduced the concept of a fuzzy topological space [Cha68], which adapts the general topology to the degree of membership rather than the degree of binary inclusion. His framework allowed elements to belong to open sets with different degrees of membership, providing a more flexible model for analyzing situations characterized by uncertainty. Chang's foundational work has opened up new methods for the application of topology in fuzzy environments and laid the groundwork for further research. However, over time, the limitations of Chang's framework became apparent, especially concerning the continuity of constant functions. Aware of these problems, Robert Lowen introduced a revised definition [Low76] in 1976 to ensure the continuity of constant functions in fuzzy topological spaces. Lowen's improvements brought fuzzy topological spaces closer to general topology, strengthened the theoretical foundations of the field, and led to its recognition as a powerful area of mathematical research.

Fuzzy topological spaces provide a framework that combines fuzzy set theory with general topology, allowing the possibility of exploring spaces where the degree of membership of each element is between 0 and 1. In this way, we can apply the extension of classical concepts such as continuity and compactness to environments characterized by degrees of membership that are either graded or partial, opening up new possibilities for analysis, and applications in uncertain or imprecise environments.

This thesis aims to systematically study the fundamentals and properties of fuzzy topological spaces, focusing on fuzzy continuous functions and compactness in this broad framework. The main objective is to contribute to the theoretical foundations of fuzzy topology and to explore its practical implications. Chapter 2 introduces the basic background of fuzzy sets and discusses the basic operations of fuzzy set theory. Chapter 3 delves into the concepts of fuzzy topology (Chang's), defines fuzzy topological spaces and examines their properties and structure.

The next chapters build on these foundations to explore advanced topics such as fuzzy continuity and compactness. Chapter 4 discusses Lowen's definition and the fuzzy continuous functions, which are an extension of classical continuity and are fundamental to understanding mappings between fuzzy topological spaces. Chapter 5 explores the concept of compactness in fuzzy topology. Finally, Chapter 6 discusses the applications and potential for the future of fuzzy mathematics.

Through a comprehensive study of these topics, we seek to deepen the understanding of fuzzy topology as a theoretical framework and tool for applied mathematics. The findings of this thesis are intended to help the wider mathematical community understand how fuzzy topological concepts can be applied to practical real-world scenarios where uncertainty plays a central role.

CHAPTER 2

FUZZY SETS

The concept of fuzzy set is fundamental to fuzzy topological space. It was first introduced by Lotfi Zadeh [Zad65] in 1964 as an extension of the classical set. Let us begin with the concept of a fuzzy set.

2.1 Fuzzy Set and Crisp Set

We begin by introducing the concept of a membership function, which is fundamental to the theory of fuzzy set and allows us to define degrees of membership for elements in X.

Definition 2.1.1. Membership Function [Zad65]
Let X be a subset of the real numbers $\mathbb{R}$, denoted as $X \subseteq \mathbb{R}$. A **membership function** $\mu(x)$ on X is any function from X to a real closed unit interval $I = [0, 1]$. Namely, for all $x \in X$

$$\mu : X \to [0, 1].$$

The value of μ at x representing the "grade of membership", quantifies the degree to all $x \in X$, with $\mu(x) = 0$ indicating no membership and $\mu(x) = 1$ representing the full membership.

Definition 2.1.2. Fuzzy Set [Zad65, ZN21]
Let $X \subseteq \mathbb{R}$, and let $\mu_A : X \to [0, 1]$ be the membership function for a set $A \subseteq X$. A **fuzzy set** A is defined as the set of ordered pairs

$$A = (X, \mu_A) = \{(x, \mu_A(x))|x \in X\}$$

where $\mu_A(x)$ represents the degree of membership of the element $x \in A$.

Example 2.1.3. Consider the closed interval $X = [0, 90]$. Define the membership function $\mu : X \to [0, 1]$ as follows:

$$\mu(x) = \frac{x}{120}, \quad \forall x \in X.$$

Then (X, μ) is a fuzzy set.

Example 2.1.4. Consider $X \subseteq \mathbb{R}$. The membership function $\mu : X \to [0, 1]$ is defined as:

$$\mu(x) = \frac{1}{1 + x^2}, \quad \forall x \in X.$$

Since $\mu(x) \in [0, 1]$ for all $x \in X$, (X, μ) is a fuzzy set.

Definition 2.1.5. Characteristic Function
Let $A \subseteq \mathbb{R}$. The **characteristic function** of the set A is a function $\chi_A : A \to \{0, 1\}$ which defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.1.6. Crisp Set [ZN21]
Let $X \subseteq \mathbb{R}$. A **crisp set** in X is a set of ordered pairs $A = (X, \chi_A)$, where $\chi_A(x)$ is the characteristic function.

Example 2.1.7. Let $X = [0, 20]$, and define $A = \{x \in X | x \in \mathbb{Q}\}$. Then for each $x \in X$, the characteristic function is given by:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then, (X, χ_A) is a crisp set.

Remark 2.1.8. Crisp set is a special subclass of fuzzy set, where the membership function $\mu(x)$ is restricted to take only the values 0 or 1, corresponding to full exclusion or inclusion in the set.

Let us look at the following definitions:

Definition 2.1.9. [Zad65]
Given a fuzzy set $A = (X, \mu_A)$ and for all $x \in X$,

(a) Two fuzzy sets A and B are said to be equal, denoted as $A = B$, if $\mu_A(x) = \mu_B(x)$.

(b) A is a subset of B, denoted as $A \subseteq B$, if $\mu_A(x) \leq \mu_B(x)$.

(c) A is an empty fuzzy set, denotes as $A = \emptyset_X$, if $\mu_A(x) = 0$.

(d) A is a fully included fuzzy set, denoted as $A = \sqcup_X$, if $\mu_A(x) = 1$.

2.2 Basic Operations of Fuzzy Sets

In this section, we will study the complement of fuzzy set and the union and intersection of fuzzy sets.

Definition 2.2.1. Complement of a Fuzzy Set [Zad65]
Given a fuzzy set $A = (X, \mu_A)$, its **complement** A^c is defined by the following membership function:

$$\mu_{A^c}(x) = 1 - \mu_A(x), \ \forall x \in X.$$

Example 2.2.2. Let $X = [0, 90]$ and define $\mu_A : X \to [0, 1]$ by

$$\mu_A(x) = \frac{x}{90}, \ \forall x \in X.$$

Then $A = (X, \mu_A)$ is a fuzzy set. The complement of A is defined as $A^c = (X, \mu_{A^c})$ is defined as

$$\mu_{A^c}(x) = 1 - \frac{x}{90}, \ \forall x \in X.$$

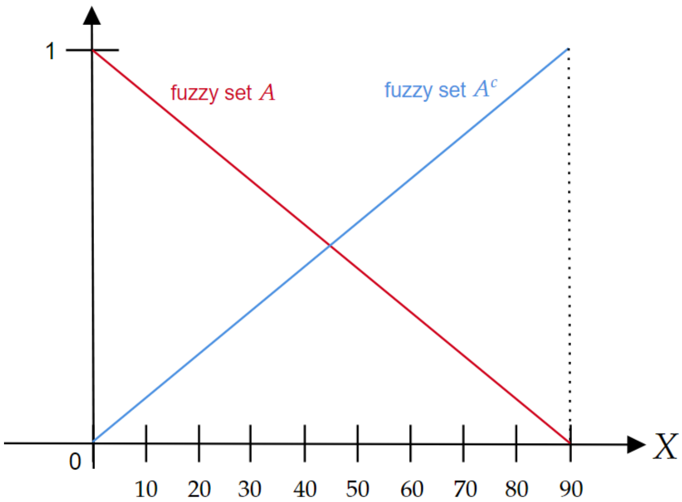

Figure 2.1: Fuzzy set A and its complement A^c.

Definition 2.2.3. Union and Intersection [Zad65]
Given an index set $\mathcal{J}$ and a family of fuzzy sets $A_j = (X, \mu_{A_j})_{j \in \mathcal{J}}$, we define

- the **union** of fuzzy sets as

$$\bigcup_{j\in\mathcal{J}} A_j = (X, \mu_{\bigcup_{j\in\mathcal{J}} A_j})$$

where

$$\mu_{\bigcup_{j\in\mathcal{J}} A_j}(x) := \sup_{j\in\mathcal{J}}\{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

In particular, when $\mathcal{J}$ is finite, then

$$\mu_{\bigcup_{j\in\mathcal{J}} A_j}(x) = \max_{j\in\mathcal{J}}\{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

- the **intersection** of fuzzy sets as

$$\bigcap_{j\in\mathcal{J}} A_j = (X, \mu_{\bigcap_{j\in\mathcal{J}} A_j})$$

where

$$\mu_{\bigcap_{j\in\mathcal{J}} A_j}(x) = \inf_{j\in\mathcal{J}}\{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

In particular, when $\mathcal{J}$ is finite, then

$$\mu_{\bigcap_{j\in\mathcal{J}} A_j}(x) = \min_{j\in\mathcal{J}}\{\mu_{A_j}(x)\} \text{ for all } x \in X.$$

Theorem 2.2.4. [Zad65]
If A and B are two fuzzy sets, then

(1) $(A \cup B)^c = A^c \cap B^c$ (by De Morgan's Laws),

(2) $(A \cap B)^c = A^c \cup B^c$,

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (by Distributive Laws),

(4) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Consider $1 - \max\{\mu_A, \mu_B\} = \min\{1 - \mu_A, 1 - \mu_B\}$ for the case $\mu_A < \mu_B$ and $\mu_A > \mu_B$. For case $\mu_A < \mu_B$, then $A \subseteq B$ and hence $1 - \mu_A > 1 - \mu_B$. This shows $B^c \subseteq A^c$. Similarly, for case $\mu_A > \mu_B$, then $B \subseteq A$ and hence $1 - \mu_B > 1 - \mu_A$. This shows $A^c \subseteq B^c$ and we have (1). Similar for (2).

For (3), we have $\max\{\mu_A, \min\{\mu_B, \mu_C\}\} = \min\{\max\{\mu_A, \mu_B\}, \max\{\mu_A, \mu_C\}\}$. It is easy to check this equality by considering six cases:

1. $\mu_A > \mu_B > \mu_C$,
2. $\mu_A > \mu_C > \mu_B$,
3. $\mu_B > \mu_A > \mu_C$,
4. $\mu_B > \mu_C > \mu_A$,
5. $\mu_C > \mu_A > \mu_B$,
6. $\mu_C > \mu_B > \mu_A$.

Similar for (4). □

CHAPTER 3

FUZZY TOPOLOGICAL SPACE

In this chapter, we will study the well-established concepts of general topology to fuzzy sets. Before we study fuzzy topological spaces, it is important to first recall the fundamentals of general topological spaces, as they serve as the basis for this generalization.

3.1 General Topological Spaces

Definition 3.1.1. General Topological Space [Mun00, Teo23]
Given a set X, a **topology** on X is a collection of $\mathscr{T}$ of subsets of X that satisfies the following properties:

(1) $\emptyset, X \in \mathscr{T}$.

(2) Any arbitrary union of members of τ belongs to $\mathscr{T}$.

(3) The intersection of finite number of members of τ belongs to $\mathscr{T}$.

If τ is a topology for X, then the pair $(X, \mathscr{T})$ is a **topological space**.

Given a topological space $(X, \mathscr{T})$, we say that a subset U of X is an **open set** of X if $U \in \mathscr{T}$.

Definition 3.1.2. Discrete Topology and Indiscrete Topology [Mun00, Teo23]
Let X be a nonempty set. Then

1. The family of all subsets of X, known as the power set $P(X)$, is a topology on X and it is defined as **discrete topology**.
2. The collection of set consisting $\emptyset$ and X is a topology on X and it is defined as **indiscrete topology**.

Definition 3.1.3. Neighbourhood [Mun00, Teo23]
Let X be a topological space and let $x \in X$. A set U is a **neighbourhood** of x if U is an open set in X that contains x.

Definition 3.1.4. Interior [Mun00, Teo23]
Given that X is a topological space and $A \subset X$, then the **interior** of A is the union of all the open sets contained in A. It is denoted as $\text{int}(A)$ or A^0.

Definition 3.1.5. Finer and Coarser [Mun00, Teo23]
Suppose $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a given set X. If $\mathscr{T}_1 \subseteq \mathscr{T}_2$, we say $\mathscr{T}_2$ is **finer** than $\mathscr{T}_1$ or $\mathscr{T}_1$ is **coarser** than $\mathscr{T}_2$. If $\mathscr{T}_1 \subset \mathscr{T}_2$, we say $\mathscr{T}_2$ is **strictly finer** than $\mathscr{T}_1$ or $\mathscr{T}_1$ is **strictly coarser** than $\mathscr{T}_2$.

Definition 3.1.6. Basis of Topology [Mun00, Teo23]
Given set X, let $\mathcal{B}$ be the collection of subsets of X. Then $\mathcal{B}$ is called a **basis** for a topology on X if $\mathcal{B}$ satisfies the following properties:

(1) For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(2) If B_1 and B_2 are elements of $\mathcal{B}$ and x is an element of X such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathscr{T}$ generated by $\mathcal{B}$ as follows: Given $U \subset X$, then $U \in \mathscr{T}$ if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Definition 3.1.7. Subbasis of Topology [Mun00, Teo23]
A **subbasis** σ for a topology on X is a collection of subsets of X whose union is equal to X. The topology generated by the subbasis σ is defined to be the collection $\mathscr{T}$ of all union of finite intersections of elements of σ.

3.2 Fuzzy Topological Spaces

Now, it is enough to study the concepts of fuzzy topological spaces.

Definition 3.2.1. Fuzzy Topological Space [Cha68, PL80a]
Given a set X, a **fuzzy topology** on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\emptyset_X, \sqcup_X \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \delta$ for all $j \in \delta$, then $\bigcup_{j \in \mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a **fuzzy topological space**, or fts in short.

Moreover, every member of δ is called an δ-**open** fuzzy set. A fuzzy set is **closed** if and only if its complement is δ-open. In the sequel, if there is no confusion likely to arise, we will simply call δ-open (closed) fuzzy set as an open (closed) set.

Same as general topology, the **indiscrete fuzzy topology** contains $\emptyset_X$ and $\sqcup_X$ only, while the **discrete fuzzy topology** contains all fuzzy sets.

Let δ, γ be two fuzzy topologies for X with $\delta \subseteq (\subset)\gamma$, then we say γ is **finer** (strictly) than δ or δ is **coarser** (strictly) than γ.

Definition 3.2.2. Basis of Fuzzy Topological Space [PL80a]
Let (X, δ) be a fts. A subfamily β of δ is called **basis** for δ if for each $A \in \delta$, there exists $\beta_A \subseteq \beta$ such that $A = \bigcup \beta_A$. A subfamily σ of δ is called a **subbasis** for δ if the family $\beta = \{\bigcap \mathcal{K} | \mathcal{K}$ is a finite subset of $\sigma\}$ is a basis for δ.

Example 3.2.3. Given two fuzzy sets on X such that μ_A, μ_B are both membership function defined on X and $X \subseteq \mathbb{R}$ such that

$$\mu_A = \begin{cases} 0, & \text{when } x \leq 0 \text{ and } x \geq 50, \\ \dfrac{x}{20}, & \text{when } 0 < x < 20, \\ 1, & \text{when } 20 \leq x \leq 30, \\ \dfrac{50 - x}{20}, & \text{when } 30 < x < 50. \end{cases}$$

And,

$$\mu_B = \begin{cases} 0, & \text{when } x \leq -20 \text{ and } x \geq 30, \\ \dfrac{20 + x}{20}, & \text{when } -20 < x < 0, \\ 1, & \text{when } 0 \leq x \leq 10, \\ \dfrac{30 - x}{20}, & \text{when } 10 < x < 30. \end{cases}$$

Then the union and intersection of two fuzzy sets are

$$\mu_{A \cup B} = \max\{\mu_A(x), \mu_B(x)\} \text{ and } \mu_{A \cap B} = \min\{\mu_A(x), \mu_B(x)\}.$$

The family of fuzzy sets $\delta = \{\mu_A, \mu_B, \mu_{A \cup B}, \mu_{A \cap B}\}$ defined on X forms a fts.

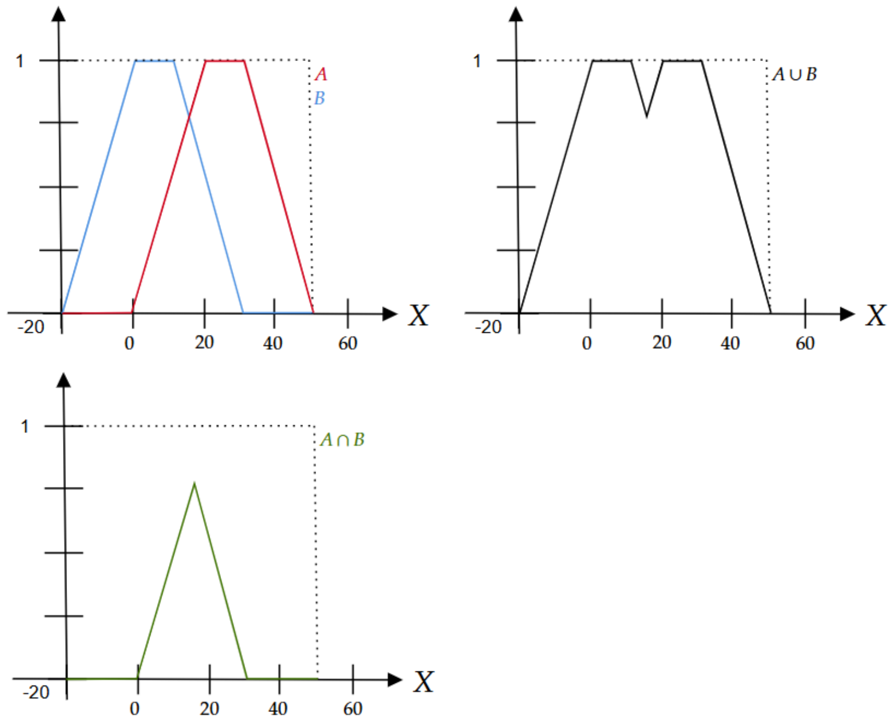

Figure 3.1: Fuzzy topology defined on X.

Definition 3.2.4. Neighbourhood [Cha68]

A fuzzy set U in a fts (X, δ) is a neighbourhood, or $nbhd$ for short, of a fuzzy set A if there exists an open fuzzy set O such that $A \subseteq O \subseteq U$.

Theorem 3.2.5. Open Fuzzy Set [Cha68]

A fuzzy set A is open if and only if for every fuzzy set B contained in A, A is a $nbhd$ of B.

Proof. Assume A is open. By definition 3.2.4, every fuzzy set B contained in A is also contained in an open set. Moreover, since $A \subseteq A$, then we have $B \subseteq O = A \subseteq A$. Hence, A is the $nbhd$ of B. Conversely, assume that for every fuzzy set B that contained in A, A is a $nbhd$ of B, since $A \subseteq A$, A is a $nbhd$ of itself. Then there is an open set O such that $A \subseteq O \subseteq A$. This shows $A = O$. Hence A is an open fuzzy set. □

Theorem 3.2.6. Intersection of Finite Neighbourhood [Cha68]

If U is a $nbhd$ system of a fuzzy set A, then the finite intersection of members of U belongs to U, and each fuzzy set that contains a member of U belongs to U.

Proof. Suppose $U = \{U_1, U_2, \ldots, U_n\}$ is a $nbhd$ system of a fuzzy set A, then there exists $O_1, O_2, \ldots, O_n$ be respectively open fuzzy set such that $A \subset O_i \subset U_i$ where $i = 1, 2, \ldots, n$. Since $\cap_{i=1}^{n} U_i$ contains $\cap_{i=1}^{n} O_i$ and it is a $nbhd$ of A, this shows $\cap_{i=1}^{k} U_i$ is a $nbhd$ system of A for $k = 2, \ldots, n$. Hence, the finite intersection of members of U belongs to U. If a fuzzy set B contains a member of U, then it contains an open $nbhd$ of A, and B is an open $nbhd$ of A. Hence, B is belongs to U. □

Remark 3.2.7. This theorem ensures that the finite intersection of open $nbhd$ of a fuzzy set A is still an open $nbhd$ of A.

Definition 3.2.8. Interior Fuzzy Set [Cha68]
Let A, B be fuzzy sets in a fts (X, δ) and let $B \subseteq A$. Then B is called an **interior fuzzy set** of A if A is a $nbhd$ of B. The union of all interior fuzzy sets of A is called the interior of A and is denoted by int(A).

Theorem 3.2.9. [Cha68]
Consider fts (X, δ). For every $A \in \delta$, int(A) is open and it is the largest open fuzzy set contained in A.

Proof. For a given fuzzy set A, let B_α be the interior fuzzy sets of A where $\alpha \in \mathcal{A}$. Let O_λ be the open fuzzy sets where $\lambda \in \Lambda$ such that $O_\lambda \subseteq A$. Let λ be the map from $\mathcal{A}$ to Λ. By definition 3.2.4 and definition 3.2.8, for every B_α there is a $O_{\lambda(\alpha)}$ such that $B_\alpha \subseteq O_{\lambda(\alpha)} \subseteq A$. Notice that the choice of $O_{\lambda(\alpha)}$ might not be unique since it is possible that $O_{\lambda(\alpha_1)} = O_{\lambda(\alpha_2)}$ for $\alpha_1 \neq \alpha_2$ where $\alpha_1, \alpha_2 \in \mathcal{A}$. In particular, every O_λ is also an interior fuzzy set of A. Since $O_\lambda \subseteq O_\lambda \subseteq A$. This shows $\{O_\lambda\}_{\lambda\in\Lambda}$ is a subcollection of $\{B_\alpha\}_{\alpha\in\mathcal{A}}$. Hence,

$$\text{int}(A) = \bigcup_{\alpha\in\mathcal{A}} B_\alpha \subseteq \bigcup_{\alpha\in\mathcal{A}} O_{\lambda(\alpha)} \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda,$$

and since $\{O_\lambda\}_{\lambda\in\Lambda}$ is a subcollection of $\{B_\alpha\}_{\alpha\in\mathcal{A}}$, it shows

$$\bigcup_{\lambda\in\Lambda} O_\lambda \subseteq \bigcup_{\alpha\in\mathcal{A}} B_\alpha = \text{int}(A).$$

Therefore,

$$\text{int}(A) = \bigcup_{\alpha\in\mathcal{A}} B_\alpha = \bigcup_{\lambda\in\Lambda} O_\lambda = \text{int}(A).$$

Since $O = \bigcup_{\lambda \in \Lambda} O_\lambda$ is the union of open fuzzy sets, it is also an open fuzzy set and contains all open fuzzy sets O_λ in A. Hence, int(A) is open and it is the largest open fuzzy set contained in A. □

Since int(A) is the largest open fuzzy set contained in A, it leads us to the following corollary.

Corollary 3.2.10. [Cha68]
The fuzzy set A is open if and only if $A =$ int(A).

Proof. If A is open, then $A \subseteq$ int(A). Since int(A) is the largest open fuzzy set contained in A, then $A =$ int(A). Conversely, assume $A =$ int(A). By theorem 3.2.9, A is an open set. □

3.3 Sequences of Fuzzy Sets

Definition 3.3.1. Convergence of Fuzzy Set [Cha68]

A sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ is said to be **eventually contained** in a fuzzy set A if there is an integer m such that, if $n \geq m$, then $A_n \subset A$. If the sequence $\{A_n\}$ is in a fts (X, δ), then we say the sequence converges to a fuzzy set A if and only if it is eventually contained in each $nbhd$ of A.

Definition 3.3.2. Cluster Fuzzy Set [Cha68]
A sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ is said to be **frequently contained** in a fuzzy set A if for each integer m there is an integer n such that $n \geq m$, and $A_n \subset A$. If the sequence $\{A_n\}$ is in fts (X, δ), then a fuzzy set A is a **cluster fuzzy set** of $\{A_n\}$ if and only if $\{A_n\}$ is frequently contained in every $nbhd$ of A.

Definition 3.3.3. [Cha68]
Let N be the map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1, 2, \ldots\}$ is a subsequence of a sequence $\{A_n, n = 1, 2, \ldots\}$ if there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$.

Theorem 3.3.4. [Cha68]
If the $nbhd$ system of each fuzzy set in a fts (X, δ) is countable, then

(a) A fuzzy set A is open if and only if each sequence of fuzzy set $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A.

(b) If A is a cluster fuzzy set of a sequence $\{A_n, n = 1, 2, \ldots\}$ of fuzzy sets, then there is a subsequence of the sequence converging to A.

Proof. (a) Assume A is open, then for every fuzzy set B that contained in A, A is a $nbhd$ of B. By definition 3.3.2, if $\{A_n\}$ is converges to B, then there is an integer $m \in \mathbb{Z}$ such that for all $n \geq m$, $A_n \subseteq B$. Since A is a $nbhd$ of B, $\{A_n\}$ is eventually contained in A.

Conversely, assume that each sequence of fuzzy set $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A. Then for each $B \subseteq A$, let $\{U_n | n = 1, 2, \ldots\}$ be the $nbhd$ system of B and let $V_n = \bigcap_{i=1}^{n} U_i$. Then $\{V_n | n = 1, 2, \ldots\}$ is a sequence of $nbhd$ that converges to B. By definition 3.3.2, it is eventually contained in each $nbhd$ of B. This shows $\exists m \in \mathbb{N}$ such that $\forall n \geq m$, $\{V_n\} \subseteq B \subseteq A$. Since $\{V_n\}$ is $nbhd$ system of B and $\{V_n\} \subseteq A$, by theorem 3.2.5, A is open.

(b) Let $\{U_n | n = 1, 2, \ldots\}$ be the $nbhd$ system of A. Then, let $S_n = \bigcup_{i=1}^{n} U_i$ such that $S_{n+1} \subseteq S_n$ for each $n \in \mathbb{N}$. This shows the sequence $\{S_n\}$ is decreasing. Then for every non-negative i, by definition 3.3.3, we may choose $N(i)$ such that $N(i) \geq i$ and $A_{N(i)} \subset S_i$. This shows $N(i)$ maps i to an index set in $\{A_n\}$ such that $A_{N(i)} \subset S_i$. Clearly, $\{A_{N(i)}\}$ is a subsequence of $\{A_n\}$. Since $\{S_n\}$ is converges to A, $\{A_{N(i)}\}$ must converges to A.

□

CHAPTER 4

FUZZY CONTINUOUS FUNCTION

The concept of continuous is very important in Mathematics. In this chapter, we will study the continuous function between a fuzzy topological space to another fuzzy topological space.

4.1 Lowen's Fuzzy Topological Space

Before we study fuzzy continuity, let us look at another definition of fuzzy topological space. Let us recall the definition of Chang's[Cha68].

Definition 4.1.1. [Cha68]
Given a set X, a fuzzy topology on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\emptyset_X, \sqcup_X \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \mathcal{F}$ for all $j \in \delta$, then $\bigcup_{j\in\mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a fuzzy topological space.

However, Lowen [Low76] pointed out that under Chang's definition, constant functions between fuzzy topological spaces are not necessarily continuous. This is a significant deviation from general topology, where constant functions are trivially continuous. Lowen supports his argument by providing concrete examples, which highlight the shortcomings of Chang's definition. To solve this issue, Lowen introduced an alternative definition that guarantees the continuity of constant functions, ensuring greater consistency with general topological concepts.

Definition 4.1.2. Lowen's Fuzzy Topological Space [Low76]
Given a set X, a fuzzy topology on X is a family $\delta = \{A_j | j \in \mathcal{J}\}$ of fuzzy sets that satisfies the following properties:

(1) $\forall$ constant fuzzy set α, $\alpha \in \delta$.

(2) Closed under finite intersection: If $A_1, A_2 \in \delta$, then $A_1 \cap A_2 \in \delta$.

(3) Closed under arbitrary union: If $A_j \in \delta$ for all $j \in \delta$, then $\bigcup_{i \in \mathcal{J}} A_i \in \delta$.

If δ is a fuzzy topology for X, then the pair (X, δ) is a fuzzy topological space.

We will use this concept of fuzzy topology throughout the sequel. For Chang's definition, we will refer to **quasi fuzzy topology**.

4.2 The Function ω and ι

Let $\mathscr{J}(X)$ be the family of all topologies on X and $\mathscr{W}(X)$ be the set of all fuzzy topologies on X. On $\mathbb{R}$, we consider the topology $\mathscr{J}_r = \{(\alpha, \infty) \cup \{\emptyset\} | \alpha \in \mathbb{R}\}$. The topological space one obtains unit interval I the induced topology on $\mathscr{J}_r$ is denoted as I_r. Then we define the following maps.

Definition 4.2.1. Topologically Generated [Low76]
For the sets $\mathscr{J}(X)$ and $\mathscr{W}(X)$, we define the mapping

$$\iota : \mathscr{W}(X) \to \mathscr{J}(X)$$
$$\delta \mapsto \iota(\delta)$$

where $\iota(\delta)$ is a initial topology on X for the family of "function" δ and the topological space I_r. Then we define the mapping

$$\omega : \mathscr{J}(X) \to \mathscr{W}(X)$$
$$\mathscr{T} \mapsto \omega(\mathscr{T})$$

where $\omega(\mathscr{J}) = \mathscr{C}(\mathscr{J}, I_r)$ is a continuous function from $(X, \mathscr{J})$ to I_r. For every $\delta \in \mathscr{W}(X)$, δ is said to be **topologically generated** if $\delta = \omega(\mathscr{T})$ for some $\mathscr{T} \in \mathscr{J}(X)$.

Remark 4.2.2.

(a) $\mathscr{J}_r$ is a topology.

- $\emptyset$ is open. $\mathbb{R} = \bigcup_{k \in \mathbb{R}} (-k, \infty)$ is open.
- Any union of open sets forms an open set. The finite intersection of open sets also forms an open set. Namely, $\bigcap_{m=1}^{n} (\alpha_m, \infty) = (\max\{\alpha_1, \ldots, \alpha_n\}, \infty)$ is an open set in $\mathbb{R}$.

(b) $I_r = \{(\alpha, 1] | \forall \alpha \in \mathbb{R}\}$ is an induced topology.

- For $\alpha \le 0$, $[0, 1]$ is an open in I_r.
- For $0 < \alpha < 1$, $(\alpha, 1)$ is an open set in I_r.
- For $\alpha \ge 1$, $\emptyset$ is an open set in I_r.

(c) $\iota(\delta) = \mathscr{T}$ is a topology.

- Fixed r where $(r, 1] \in I_r$. For every $\alpha \in \delta$, $\iota(\alpha) = \begin{cases} \emptyset, & \text{if } \alpha \ge r, \\ X, & \text{if } \alpha < r. \end{cases}$
- $\iota(\bigcup_i \delta_i) = \bigcup_i \iota(\delta_i)$ and $\iota(\bigcap_i^n \delta_i) = \bigcap_i^n \iota(\delta_i)$ are both open set in $\mathscr{T}$.

Hence, we can conclude for every $\delta \in \mathscr{W}(X)$,

$$\iota(\delta) = \{\iota_r(\mu) \,|\, \forall r \in [0, 1],\, \forall \mu \in \delta\} \text{ where } \iota_r(\mu) = \{x \in X, \mu(x) > r\}.$$

Hence $\iota_r : \delta \to \iota_r(\delta)$ is indeed a continuous function since it maps from an open set in δ to another open set in $\mathscr{T}$.

(d) Notice that $\omega(\mathscr{T})$ is a continuous function from $(X, \mathscr{T})$ to I_r if and only if for every $r \in [0, 1]$,

$$\mu^{-1}((r, 1]) = \{x \in X \,|\, \mu(x) > r\} \text{ is open in } X.$$

Moreover, $\omega(\mathscr{T})$ is a lower semicontinuous function from $(X, \mathscr{T})$ to I which equipped by Euclidean topology for every $\mathscr{T} \in \mathscr{J}(X)$.

Proposition 4.2.3. [Low76]

(1) $\iota \circ \omega = id_{\mathscr{T}(X)}$.

(2) ι and ω are respectively an isotone surjection and isotone injection.

(3) $\omega \circ \iota(\delta)$ is the smallest topologically generated fuzzy topology which contains δ and it is denoted as $\bar{\delta}$.

(4) δ is topologically generated if and only if $\delta = \bar{\delta}$.

Proof. (1) Notice that $\omega(\mathscr{J}) = \mathscr{C}(X, I_r)$. By remark (c) and (d), we have

$$\iota(\omega(\mathscr{J})) = id : \mathscr{J}(X) \to \mathscr{J}(X).$$

(2) Notice that if $\mathscr{T}_1 \subseteq \mathscr{T}_2$, then $\mathscr{T}_2$ is a refinement of $\mathscr{T}_1$.

For ι, if $\delta_1 \subseteq \delta_2$, then δ_2 has more open fuzzy set than δ_1. Then $\iota(\delta_2)$ has more open set than $\iota(\delta_1)$. This shows $\iota(\delta_1) \subseteq \iota(\delta_2)$. This shows ι is an isotone map. Now, for every $\mathscr{T} \in \mathscr{J}$, $\omega(\mathscr{T}) \in \mathscr{W}(X)$ is the element such that $\iota(\omega(\mathscr{T})) = \mathscr{T}$. Hence, ι is an isotone surjection.

For ω, for every $\mathscr{T}_1, \mathscr{T}_2 \in \mathscr{J}$, if $\mathscr{T}1 \subseteq \mathscr{T}_2$, then $\omega(\mathscr{T}_1) \subseteq \omega(\mathscr{T}_2)$. Hence, ω is an isotone map. Assume $\omega(\mathscr{T}_1) = \omega(\mathscr{T}_2)$, then we have

$$\iota\big(\omega(\mathscr{T}_1)\big) = \iota\big(\omega(\mathscr{T}_2)\big) \Leftrightarrow \mathscr{T}_1 = \mathscr{T}_2$$

Hence, ω is an isotone injection.

(3) For every $\delta \in \mathscr{W}(X)$, $\iota(\delta) \in \mathscr{J}(X)$ is a topology on X. Hence, $\omega(\iota(\delta))$ is a topologically generated fuzzy topology. Now, we want to show $\bar{\delta} = \omega(\iota(\delta))$ is the smallest topologically generated fuzzy topology that contains δ.

Assume $\Delta = \omega(\mathscr{T})$ is a topologically generated fuzzy topology for some $\mathscr{T} \in \mathscr{J}(X)$. If $\delta \subseteq \Delta$, we want to show $\bar{\delta} \subseteq \Delta$. Then, we have

$$\delta \subseteq \omega(\mathscr{T}) = \Delta$$
$$\iota(\delta) \subseteq \iota(\omega(\mathscr{T})) = \mathscr{T}$$
$$\bar{\delta} = \omega\big(\iota(\delta)\big) \subseteq \omega(\mathscr{T}) = \Delta$$

This shows $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ. Moreover, $\bar{\delta} = \bigcap_{\delta \subseteq \Delta} \Delta$ where Δ is topologically generated.

(4) If δ is topologically generated, then $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ. Hence $\bar{\delta} = \delta$. Conversely, if $\delta = \bar{\delta}$, then δ is topologically generated.

□

Theorem 4.2.4. [Low76]
(X, δ) is topologically generated if and only if for each continuous function $f \in \mathscr{C}(I_r, I_r)$ and for each $\nu \in \delta$, $f \circ \nu \in \delta$.

Proof. Assume (X, δ) is topologically generated. Since $\nu \in \mathscr{C}(\mathscr{T}, I_r) = \delta$ and $f \in \mathscr{C}(I_r, I_r)$, then $f \circ \nu \in \mathscr{C}(\mathscr{T}, I_r) = \delta$.

Conversely, assume $\mu \in \bar{\delta}$. Recall that $\bar{\delta} = \omega \circ \iota(\delta)$. This shows $\mu \in \mathscr{C}\big(\iota(\delta), I_r\big)$. Since a basis for $\iota(\delta)$ is provided by the finite intersections

$$\bigcap_{i=1}^{n} \nu_i^{-1}\big((r_i, 1]\big) \quad \text{for some } \nu_i \in \delta,\ r_i \in I;$$

this is equivalent to saying for any $r \in I$, any $x \in \mu^{-1}\big((r,1]\big)$, $(r,1]$ is open in I_r and $\mu^{-1}(r,1]$ is open in $\iota(\delta)$ since $\mu \in \mathscr{C}\big(\iota(\delta), I_r\big)$. Hence, for every $x \in \mu^{-1}(r,1]$, there exists finite open set $(r_i, 1]$ such that

$$x \in \bigcap_{i \in I_{r,x}} \nu_i^{-1}\big((r_i,1]\big) \subseteq \mu_i^{-1}\big((r_i,1]\big).$$

Now, we want to show μ is closed under some finite intersection and arbitrary union of basis of δ. Fix x and let $\mu(x) = k_x \in (r,1]$, then $\forall x < k_x$, $\exists$ a finite index set I_r such that

$$x \in \bigcap_{i \in I_r} \nu_i^{-1}\big((r_i,1]\big) \subseteq \mu_i^{-1}\big((r_i,1]\big).$$

Then, $\forall r < k_x$ and $\forall i \in I_r$, let

$$\mu_{i,r}(y) = \big((r\chi_{(i_i,1]}) \circ \nu_i\big)(y) = \begin{cases} r, \text{ if } \nu_i(y) > r_i, \\ 0, \text{ if } \nu_i(y) \le r_i. \end{cases}$$

where $\mu_{i,r} \in \delta$ and $r\chi_{(i_i,1]} \in \mathscr{C}(I_r, I_r)$. This indeed follows from our assumption $f \circ \nu$ for all $f \in \mathscr{C}(I_r, I_r)$ and $\nu \in \delta$. Then, let $\nu_r^x = \inf_{i \in I_r}\{\mu_{i,r}\} \in \delta$. Since I_r is finite and hence $\nu_r^x = \min_{i \in I_r}\{\mu_{i,r}\}$, then we have

$$\nu_r^x(y) = \begin{cases} r, \text{ if } \ \forall i \in I_r \text{ we have } \nu_i(y) > r_i, \\ 0, \text{ if } \ \exists j \in I_r \text{ such that } \nu_i(y) \le r_j. \end{cases}$$

Hence if $\nu_r^x(y) = r$, then $\nu_i(y) > r_i$ for all $i \in I_r$. Since

$$y \in \bigcap_{i=1}^{n} \nu_i^{-1}(r_i, 1] \subseteq \mu^{-1}(r, 1],$$

therefore for every $y \in [0,1]$ we have $\mu(y) > r$. This shows $\mu \ge \nu_r^x$ for every $x \in \mu^{-1}(r,1]$ and $r < k_x$. Now, it is easy to see that

$$\mu = \sup_{x \in X} \sup_{r < k_x} \nu_r^x(y) \in \delta.$$

Hence, if every $\mu \in \bar{\delta}$, then $\mu \in \delta$. This shows $\bar{\delta} = \delta$ which implies (X, δ) is topologically generated. □

4.3 Function Between Two Fuzzy Sets

We first look at the function between two fuzzy sets. This will lead us to study the function between two fts.

Definition 4.3.1. Function Between Two Fuzzy Sets [Cha68, PL80b]
Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(x)$ for all x in X. Then the inverse of B, denoted as $f^{-1}[B]$, is a fuzzy set in X whose membership function is given by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \text{ for all } x \in X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$ for all x in X. Then, the image of A, denoted as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f[A]}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all x in X and y in Y where $f^{-1}(y) = \{x | f(x) = y\}$.

Clearly, since $f : X \to Y$ is well-defined, these two equalities are true for all x in X. Now, let us prove some theorem.

Theorem 4.3.2. [Cha68, PL80b]
Let f be a function from X to Y. Then,

(a) $f^{-1}[B^c] = (f^{-1}[B])^c$ for all $B \in Y$.

(b) $(f[A])^c \subseteq f[A^c]$ for all $A \in X$.

(c) If $B_1 \subseteq B_2$, then $f^{-1}[B_1] \subseteq f^{-1}[B_2]$ for all $B_1, B_2 \in Y$.

(d) If $A_1 \subseteq A_2$, then $f[A_1] \subseteq f[A_2]$ for all $A_1, A_2 \in X$.

(e) $f[f^{-1}[B]] \subseteq B$ for all $B \in Y$.

(f) $A \subseteq f^{-1}[f[A]]$ for all $A \in X$.

(g) Let $f : X \to Y$ and $g : Y \to Z$. Then, $(g \circ f)^{-1}[C] = f^{-1}(g^{-1}[C])$ for all $C \in Z$, where $g \circ f$ is the composition of g and f.

Proof. (a) For all fuzzy sets B in Y, by definition $4.1.2$, we have

$$\begin{aligned}\mu_{f^{-1}[B^c]}(x) &= \mu_{B^c}(f(x)) \\ &= 1-\mu_B(f(x)) \\ &= 1-\mu_{f^{-1}[B]}(x) \\ &= \mu_{f^{-1}[B]^c}(x) \quad \forall x \in X.\end{aligned}$$

Then, we have proved the equality.

(b) For all fuzzy sets A in X, let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then we have

$$\begin{aligned}\mu_{f[A]^c}(y) &= 1-\mu_{f[A]}(y)) \\ &= 1-\sup_{x\in f^{-1}(y)}\{\mu_A(x)\} \quad \forall x \in X.\end{aligned}$$

And,

$$\begin{aligned}\mu_{f[A^c]}(y) &= \sup_{x\in f^{-1}(y)}\{\mu_{A^c}(x)\} \\ &= \sup_{x\in f^{-1}(y)}\{1-\mu_A(x)\} \\ &= 1-\inf_{x\in f^{-1}(y)}\{\mu_A(x)\} \quad \forall x \in X.\end{aligned}$$

This shows $\mu_{f[A]^c}(y) \leq \mu_{f[A^c]}(y)$ for all $y \in Y$.

(ii) If $f^{-1}(y) = \emptyset$, by definition we have $\mu_{f[A]}(y)) = 0$. Then $\mu_{f[A]^c}(y) = \mu_{f[A^c]}(y) = 1$.

We has concluded that $\mu_{f[A]^c}(y) \leq \mu_{f[A^c]}(y)$ for all $y \in Y$. and thus we have the inequality.

(c) For all fuzzy sets B_1, B_2 in Y where $B_1 \subseteq B_2$, then we have $\mu_{B_1}(f(x)) \leq \mu_{B_2}(f(x))$ for all $x \in X$. Since $\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$, then $\mu_{f^{-1}[B_1]}(x) \leq \mu_{f^{-1}[B_2]}(x)$. Then we have proved the inequality.

(d) For all fuzzy sets A_1, A_2 in X where $A_1 \subseteq A_2$, then we have $\mu_{A_1}(x) \leq \mu_{A_2}(x)$ for all $x \in X$. Let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then $\mu_{f[A]}(y) = \sup_{x\in f^{-1}(y)}\{\mu_A(x)\}$. Then, we have $\mu_{f[A_1]}(y) \leq \mu_{f[A_2]}(y)$ for all $y \in Y$.

(ii) If $f^{-1}(y) = \emptyset$,$\mu_{f[A_1]}(y) = \mu_{f[A_2]}(y) = 0$ for all $y \in Y$.

Then, we have proved the inequality.

(e) For all fuzzy sets B in Y, let us consider two cases:

(i) If $f^{-1}(y) \neq \emptyset$, then we have

$$\begin{aligned}\mu_{f[f^{-1}[B]]}(y) &= \sup_{x \in f^{-1}(y)} \{\mu_{f^{-1}[B]}(x)\} \\ &= \sup_{x \in f^{-1}(y)} \{\mu_B(f(x))\} \\ &= \mu_B(y) \qquad \text{for all } y \in Y.\end{aligned}$$

(ii) If $f^{-1}(y) = \emptyset$, then $\mu_{f^{-1}[f[B]]}(y) = 0$ for all $y \in Y$. This shows $\mu_{f^{-1}[f[B]]}(y) \leq \mu_B(y)$ for all $y \in Y$.

Then, we have proved the inequality.

(f) For all fuzzy sets A in X,

$$\begin{aligned}\mu_{f^{-1}[f[A]]}(x) &= \mu_{f[A]}(f(x)) \\ &= \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \quad \text{for all } x \in X.\end{aligned}$$

This shows $\mu_{f^{-1}[f[A]]}(x) \geq \mu_A(x)$ for all $x \in X$. Hence, we have proved the inequality.

(g) Let f be a function from X to Y and g be a function from Y to Z. Let C be a fuzzy set in U_3, then

$$\begin{aligned}\mu_{(g\circ f)^{-1}[C]}(x) &= \mu_C(g(f(x))) \\ &= \mu_{g^{-1}[C]}(f(x)) \\ &= \mu_{f^{-1}(g^{-1}[C])}(x) \quad \text{for all } x \in X.\end{aligned}$$

Then, we have proved $(g \circ f)^{-1}[C] = f^{-1}(g^{-1}[C])$ for all $C \in Z$.

□

4.4 Continuity of Fuzzy Topological Spaces

Definition 4.4.1. Fuzzy Continuous [Cha68, PL80b, Low76]
A function $\tilde{f}$ from a fts $U_1 = (X, \delta)$ to another fts $U_2 = (Y, \gamma)$ is said to be **fuzzy continuous**,

or F-continuous in short, if the inverse of each γ-open fuzzy set is an δ-open fuzzy set. Namely, $\tilde{f}$ is F-continuous if and only if

$$\mu_{f^{-1}[\nu]} \in \delta \quad \forall \nu \in \gamma.$$

Corollary 4.4.2. [Cha68, PL80b]
If $\tilde{f}$ is a function from a fts $U_1 = (X, \delta)$ to another fts $U_2 = (Y, \gamma)$ and $\tilde{g}$ is a function from $U_2 = (Y, \gamma)$ to $U_3 = (Z, \lambda)$ where are both F-continuous, then the composition of the functions $\tilde{g} \circ \tilde{f}$ is also F-continuous.

Proof. If $\tilde{f} : U_1 \to U_2$ and $\tilde{g} : U_2 \to U_3$ are both F-continuous, then

$$(\tilde{g} \circ \tilde{f})^{-1}[U_3] = \tilde{f}^{-1}(\tilde{g}^{-1}[U_3]).$$

For every $\mu \in \delta$, $\nu \in \gamma$ and $\eta \in \lambda$, since $\tilde{g}^{-1}[\eta]$ is γ-open and $\tilde{f}^{-1}[\nu]$ is δ-open, hence $(\tilde{g} \circ \tilde{f})^{-1}[\eta] = \tilde{f}^{-1}(\tilde{g}^{-1}[\eta])$ is δ-open. Hence, we have proved the composition of F-continuous function is still a F-continuous function. $\square$

Let us study an example.

Example 4.4.3.
Consider two fts $U = (X, \delta)$ and $V = (Y, \gamma)$ and two fuzzy sets: $A \in \delta$ and $B \in \gamma$. Then, we define a function $f : X \to Y$ such that $f(x) = 4$ for all $x \in X$. By definition 4.3.1, $A = f^{-1}[B]$ is a fuzzy set in X with the following membership function:

$$\mu_{f^{-1}[B]}(x) = \mu_B(4) = \mu_B\big(f(x)\big) \text{ for all } x \in X.$$

Notice that $f : X \to Y$ is a constant function. Under Chang's definition, $f^{-1}[B]$ is not an open fuzzy set since only $\emptyset_X$ and $\sqcup_X$ are guaranteed to be part of the fuzzy topology. Therefore, Lowen introduced a refined definition to solve this continuity problem based on Chang's definition.

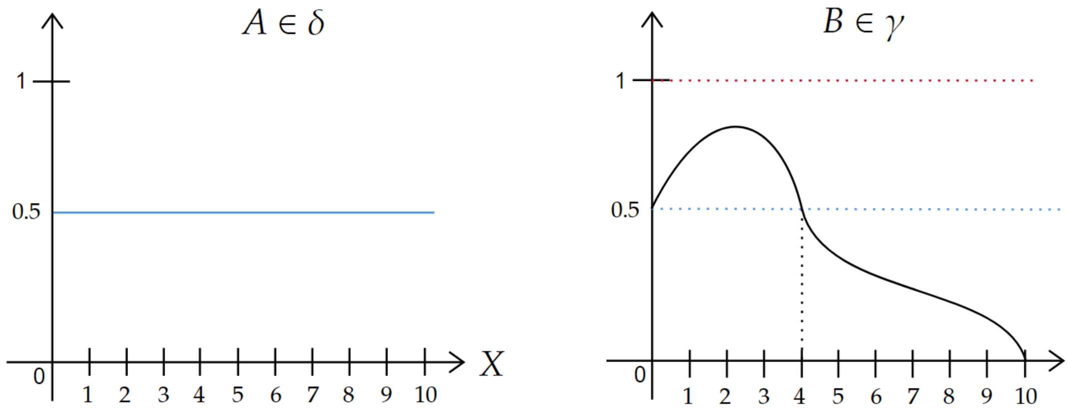

Figure 4.1: Fuzzy set $A \in \delta$ and $B \in \gamma$.

Definition 4.4.4. [Low76]
A function $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is said to be continuous if $\tilde{f} : \big(X, \iota(\delta)\big) \to \big(Y, \iota(\gamma)\big)$ is continuous. Namely,

$$\tilde{f} \in \mathscr{C}\big((X, \mathscr{T}), (Y, \mathscr{P})\big)$$

where $\mathscr{T} = \iota(\delta)$ and $\mathscr{P} = \iota(\gamma)$.

Remark 4.4.5. If δ and γ is topologically generated, then $\delta = \omega(\mathscr{T})$ and $\gamma = \omega(\mathscr{P})$ for some $\mathscr{T} \in \mathscr{J}(X)$ and $\mathscr{P} \in \mathscr{J}(Y)$. Then we have $\iota(\omega(\mathscr{T})) = \mathscr{T}$ and $\iota(\omega(\mathscr{P})) = \mathscr{P}$. This leads us to study the continuous property between general topologies and fuzzy topologies later.

Proposition 4.4.6. [Low76]
Consider the following properties for $\tilde{f} : (X, \delta) \to (Y, \gamma)$:

(1) $\tilde{f}$ is F-continuous.

(2) $\tilde{f}$ is continuous.

(3) $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous.

(4) $\tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous.

then we have $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

Proof.

- (1) $\Rightarrow$ (2)
  Assume a basis for $\iota(\delta)$ and $\iota(\gamma)$ are respectively given by $\bigcap_{i=1}^{n} \mu_i^{-1}(r_i, 1]$ and $\bigcap_{j=1}^{n} \nu_j^{-1}(r_j, 1]$ where $\mu_i \in \delta$ and $\nu_j \in \gamma$. Recall that for $f : X \to Y$,

$$\mu_i(x) = f^{-1}\big(\nu_j(x)\big) = \nu_j\big(f(x)\big) \ \ \forall x \in X.$$

  If $\tilde{f}$ is F-continuous, then we have

$$f^{-1}(\nu_j) = \nu_j\big(f(x)\big) = \mu_i(x) \in \delta \ \ \forall \nu_j \in \gamma.$$

  For each $y \in \nu_j^{-1}(r_j, 1]$ for some j, we have $\nu_j(y) \in (r_j, 1]$. Since $\nu_j^{-1}(r_j, 1]$ is an open set in $\big(Y, \iota(\gamma)\big)$, then we have

$$\mu_i(x) = f^{-1}(\nu_j) = \nu_j\big(f(x)\big) \in (r_j, 1] \Rightarrow x \in \mu_i^{-1}(r_j, 1] \ \ \forall x \in X$$

  where $\mu_i^{-1}(r_j, 1]$ is an open set in $\big(X, \iota(\delta)\big)$. Hence $\tilde{f}$ is continuous.

- (2) $\Leftrightarrow$ (3)
  Assume $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is continuous. Since $\omega = \mathscr{C}(\mathscr{J}, I_r)$, then $\tilde{\mu} \in \mathscr{C}\big((X, \mathscr{T}), I_r\big)$ and $\tilde{\nu} \in \mathscr{C}\big((Y, \mathscr{P}), I_r\big)$ are both continuous. Since $\tilde{f}$ is continuous, then we have

$$\tilde{\mu}_i^{-1}(r_j, 1] = f^{-1}\big(\tilde{\nu}_j^{-1}(r_j, 1]\big) \in \mathscr{T}$$

  for every $\tilde{\nu}_j(r_j, 1] \in \bar{\gamma}$. Since $\tilde{\mu}_i \in \mathscr{C}\big((X, \mathscr{T}), I_r\big)$, then $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Conversely, assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Notice that $\omega\big(\iota(\gamma)\big) = \bar{\gamma}$. Since $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous and $\bar{\delta}$ is the smallest topologically generated fuzzy topology that contains δ, hence $\tilde{f}$ is continuous.

- (3) $\Leftrightarrow$ (4)
  Assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous. Then we define the function g and $\tilde{g}$ such that $g : Y \to Y$ and $\tilde{g} : (Y, \bar{\gamma}) \to (Y, \gamma)$. Notice that $\bar{\gamma}$ is the smallest topologically generated fuzzy topology that contains γ, namely

$$\nu \in \bar{\gamma} \ \ , \forall \nu \in \gamma.$$

  Hence $\tilde{g}$ is F-continuous. Since $\tilde{f}$ and $\tilde{g}$ are both F-continuous, thus $\tilde{g} \circ \tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous. Conversely, assume $\tilde{f} : (X, \bar{\delta}) \to (Y, \gamma)$ is F-continuous. Since (1) $\Rightarrow$ (2), thus $\tilde{f} : \big(X, \iota(\bar{\delta})\big) \to \big(Y, \iota(\gamma)\big)$ is continuous. By (2) $\Rightarrow$ (3), $\tilde{f} : (X, \bar{\delta}) \to (Y, \bar{\gamma})$ is F-continuous.

□

Corollary 4.4.7. [Low76]

Let $\mathscr{C}(X,Y)$ be the set of all continuous function from (X,δ) to (Y,γ) and $\mathscr{C}_\omega(X,Y)$ be the set of all F-continuous function from (X,δ) to (Y,γ).If δ is topologically generated, then

$$\mathscr{C}(X,Y) = \mathscr{C}_\omega(X,Y).$$

Proof. By proposition 4.4.6, we know $\tilde{f} : (X,\bar{\delta}) \to (Y,\gamma)$ is F-continuous if and only if $\tilde{f} : (X,\delta) \to (Y,\gamma)$ is continuous. If $\delta = \bar{\delta}$, then we have $\mathscr{C}(X,Y) = \mathscr{C}_\omega(X,Y)$. □

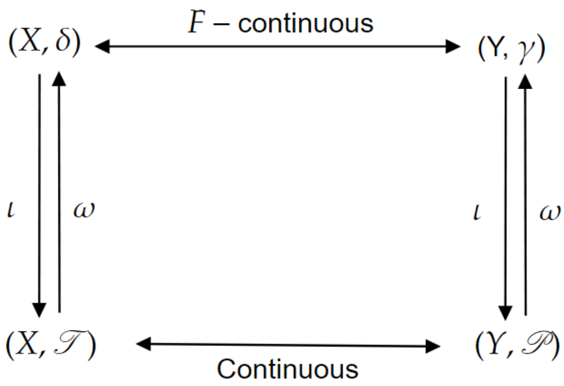

Figure 4.2: If $\delta = \bar{\delta}$, then we have the following commutative diagram.

Example 4.4.8. The inverse of corollary 4.4.7 is not true.

Consider two fts (X,δ) and (Y,γ) where $X = I = [0,1]$ and Y is arbitrary. Let δ be the fuzzy topology on X with the subbasis

$$\{\text{constant } \alpha \,|\, \forall \alpha\} \cup \{y = x \,|\, \forall x \in [0,1]\}$$

and let γ be the discrete fuzzy topology on Y, i.e., $\gamma = I^Y$. Since γ is a discrete fuzzy topology, thus $\iota(\gamma)$ is discrete. Since δ is generated by the subbasis $\{\text{constant } \alpha \,|\, \forall \alpha\} \cup \{y = x \,|\, \forall x \in [0,1]\}$, then $\iota(\delta)$ is connected. Since $\iota(\delta)$ is connected and $\iota(\gamma)$ is discrete, any function maps $\iota(\delta)$ to $\iota(\gamma)$ must be a constant function. Hence we have

$$\mathscr{C}(X,Y) = \{\text{constant function from } \iota(\delta) \text{ to } \iota(\gamma)\}.$$

By proposition 4.4.6, we have $\mathscr{C}_\omega(X,Y) \subset \mathscr{C}(X,Y)$. Since all constant functions are F-continuous, thus $\mathscr{C}_\omega(X,Y) = \mathscr{C}(X,Y)$. However, since $\iota(\delta) = \mathscr{T}_{I_r} = \{[\alpha,1] \,\forall \alpha \in [0,1]\}$ and $\omega\big(\iota(\delta)\big)$ is finer than δ, hence δ is not topologically generated.

Now, let us look at some theorem.

Theorem 4.4.9. [Cha68, PL80b]
Given $\tilde{f}$ is a function from $fts\ U_1 = (X, \delta)$ to $fts\ U_2 = (Y, \gamma)$, then we have the following statements and their relations: $(a) \Leftrightarrow (b), (c) \Leftrightarrow (d), (a) \Rightarrow (c)$ and $(d) \Rightarrow (e)$ where

(a) The function $\tilde{f}$ is F-continuous.

(b) The inverse of every fuzzy closed set is closed.

(c) For each fuzzy set A in U_1, the inverse of every $nbhd$ of $\tilde{f}[A]$ is a $nbhd$ of A.

(d) For each fuzzy set A in U_1 and each $nbhd\ V$ of $f[A]$, there is a $nbhd\ W$ of A such that $f[W] \subseteq V$.

(e) For each sequence of fuzzy sets $\{A_n | n = 1, 2, \ldots\}$ in U_1 which converges to a fuzzy set A in U_1, the sequence $\{f[A_n] | n = 1, 2, \ldots\}$ converges to $f[A]$.

Proof. (i) $(a) \Leftrightarrow (b)$

If $\tilde{f} : U_1 \to U_2$ is F-continuous and B is a closed fuzzy set in U_2, then $f^{-1}[B^c]$ is open. Since $f^{-1}[B^c] = (f^{-1}[B])^c$, thus $f^{-1}[B]$ is a closed fuzzy set. Conversely, let B be a closed fuzzy set in (Y, γ). By assumption, $f^{-1}[B]$ is closed. Since $f^{-1}[B^c] = (f^{-1}[B])^c$ and B^c are both open fuzzy sets, therefore $\tilde{f}$ is F-continuous.

(ii) $(a) \Rightarrow (c)$

If $\tilde{f}$ is F-continuous and A is a fuzzy set in U_1, let V be the $nbhd$ of $f[A]$. Then, V contains an open fuzzy set O such that $f[A] \subseteq O \subseteq V$. Then we have $f(f^{-1}[A]) \subseteq f[O] \subseteq f[V]$. Since f is F-continuous, therefore $f^{-1}[O]$ is open. By definition 3.2.1, $f^{-1}[V]$ is a $nbhd$ of A.

(iii) $(c) \Leftrightarrow (d)$

For every $A \in U_1$, if V is a $nbhd$ of $f[A]$ and $f^{-1}[V]$ is $nbhd$ of A, then there exists an open fuzzy set W such that $A \subseteq W \subseteq f^{-1}[V]$. By theorem 3.2.5 W is open if and only if it is a $nbhd$ of A. Hence W is a $nbhd$ of A and $f[W] \subseteq V$. Conversely, since $A \subseteq f^{-1}(f[A])$, then $A \subseteq W \subseteq f^{-1}(f[W]) \subseteq f^{-1}[V]$. By our assumption, $f^{-1}[V]$ is a $nbhd$ of A.

(iv) $(d) \Rightarrow (e)$

For each fuzzy set $A \in U_1$ and every $nbhd\ V$ of $f[A]$,there is a $nbhd\ W$ of A such that

$f[W] \subseteq V$. By definition 3.2.4, the sequence of fuzzy sets $\{A_n | n = 1, 2, \ldots\} \in U_1$ which converges to A is eventually contained in each $nbhd$ of A. Then there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $A_n \subseteq W$. For all $n \geq m$, since $f[A_n] \subseteq f[W] \subseteq V$, then $f[A_n]$ is eventually contained in every $nbhd$ of $f[A]$. This shows $\{f[A_n] | n = 1, 2, \ldots\}$ is converges to $f[A]$.

□

Now, we are able to define fuzzy homeomorphism.

Definition 4.4.10. Fuzzy Homeomorphism [Cha68]
A **fuzzy homeomorphism** is an F-continuous one-to-one map $\tilde{f}$ of a fts $U_1 = (X, \delta)$ onto another fts $U_2 = (Y, \gamma)$ such that the inverse of the map is also F-continuous. Then, we said U_1 is **F-homeomorphic** to U_2, or U_1 is **F-topologically equivalent** to U_2, if there is a fuzzy homeomorphism $\tilde{f} : U_1 \to U_2$.

Remark 4.4.11. We can consider the category of fuzzy topological spaces and fuzzy continuous mapping in the same way as the category of general topological spaces and continuous mapping. Moreover, the functions ω and ι induce two covariant functors between these two categories. Let $\mathscr{G}$ be the category of general topological spaces and $\mathscr{F}$ be the category of fuzzy topological spaces, then we define

$$\tilde{\omega} : \mathscr{G} \to \mathscr{F} \text{ where } \tilde{\omega}(X, \mathscr{T}) = \big(X, \omega(\mathscr{T})\big), \tilde{\omega}(\tilde{f}) = \tilde{f} \text{ and}$$
$$\tilde{\iota} : \mathscr{F} \to \mathscr{G} \text{ where } \tilde{\iota}(X, \delta) = \big(X, \iota(\delta)\big), \tilde{\iota}(\tilde{f}) = \tilde{f}.$$

By corollary 4.4.7, $\tilde{\omega}(\mathscr{G})$ is a full subcategory of $\mathscr{F}$.

CHAPTER 5

COMPACT FUZZY TOPOLOGICAL SPACES

In this chapter, we will study the compactness on quasi fuzzy topological space. Let us recall the concept of compactness from general topological spaces.

5.1 Compactness of General Topological Spaces

Definition 5.1.1. Cover and Subcover in General Topological Space [Mun00, Teo23]
In general topology, a collection $\mathcal{A}$ of subsets of a topological space $(X, \mathscr{T})$ is said to cover X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equals to X. Namely,

$$X \subseteq \bigcup_{A \in \mathcal{A}} A$$

If there is a subcollection of $\mathcal{A}$ also covers X, then we will call this subcollection as a **subcover**.

Definition 5.1.2. Open Covering [Mun00, Teo23]
For a topological space $(X, \mathscr{T})$, a covering $\mathcal{A}$ is called an **open covering** of X if every member of $\mathcal{A}$ is an open subset of X.

Definition 5.1.3. Compact Space [Mun00, Teo23]
A topological space $(X, \mathscr{T})$ is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Corollary 5.1.4. Let $a, b \in \mathbb{R}$ with $a < b$, then the closed interval $[a, b]$ is compact.

Proof. This corollary is general. The proof can be found in Munkres's book, Topology, Chapter 3, Section 27, Theorem 27.1, Page 172-173 [Mun00]. □

Definition 5.1.5. Finite Intersection Property [Mun00, Teo23]
Let X be a topological space, and let $\mathcal{C} = \{C_\alpha | \alpha \in J\}$ be a collection of subsets of X. Then, $\mathcal{C}$ has the **finite intersection property** if $\alpha_1, \ldots, \alpha_n$ are finitely many elements of J, the intersection $\bigcap_{k=1}^{n} C_{\alpha_k} \neq \emptyset$.

Theorem 5.1.6. [Mun00, Teo23]
Let X be a topological space. Then the following statements are equivalent:

(a) X is compact.

(b) If $\mathcal{C} = \{C_\alpha | \alpha \in J\}$ is a collection of closed subset of X that has finite intersection property, then $\bigcap_{\alpha \in J} C_\alpha \neq \emptyset$.

Proof. This theorem is general. The proof can be found in Munkres's book, Topology, Chapter 3, Section 26, Theorem 26.9, Page 169-170 [Mun00]. □

Theorem 5.1.7. Alexander Subbasis Theorem [Müg20]
Let $(X, \mathscr{T})$ be a topological space. If X has subbasis $\mathscr{B}$ such that every cover $\mathcal{A} = \{B_\alpha \mid \alpha \in \mathscr{B}\}$of X by elements of $\mathscr{B}$ has a finite subcover, then X is compact.

Proof. This theorem is general. The proof can be found in Mueger's book, Topology for the Working Mathematician, Chapter 7, Page 128 [Müg20]. □

5.2 Compactness of Quasi Fuzzy Topological Spaces

In this section, we will discuss the compact structure of the fuzzy topology. Similar to the general topology, we first define the open covering of fuzzy space.

Definition 5.2.1. Cover and Subcover in Fuzzy Space [Cha68]
Let $\mathcal{A}$ be a collection of fuzzy sets on X. For a fuzzy set B on X, $\mathcal{A}$ is said to be a **covering** of B if

$$B \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Moreover, this is equivalent to

$$\sup_{A \in \mathcal{A}} \{\mu_A(x)\} \geq \mu_B(x) \quad \forall x \in X.$$

A **subcover** $\mathcal{A}'$ of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which ia also a covering.

Definition 5.2.2. Open Covering in Fuzzy Topological Space [Cha68]
For a fuzzy set B on X and a (or quasi) fts $U = (X, \delta)$, a covering $\mathcal{A}$ is called an **open covering** of B if every member of $\mathcal{A}$ is an open fuzzy set, or $\mathcal{A} \subset \delta$.

Definition 5.2.3. Quasi Fuzzy Compactness [Cha68]
A quasi fts $U = (X, \delta)$ is said to be **quasi fuzzy compact**, or quasi F-compact in short, if each open covering $\mathcal{A}$ of $\sqcup_X$ has a finite subcover $\mathcal{A}'$.

Remark 5.2.4. The following are equivalent:

- A fts $U = (X, \delta)$ is quasi F-compact.
- For every open covering $\mathcal{A}$ of $\sqcup_X$ such that

$$\bigcup_{A \in \mathcal{A}} A = \sqcup_X,$$

  there is an finite open subcover $\mathcal{A}'$ of $\sqcup_X$ such that

$$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

- For every open covering $\mathcal{A}$ of $\sqcup_X$ such that

$$\sup_{A \in \mathcal{A}} \{\mu_A(x)\} = 1 \quad \forall x \in X,$$

  there is an finite open subcover $\mathcal{A}'$ of $\sqcup_X$ such that

$$\max_{A \in \mathcal{A}'} \{\mu_A(x)\} = \sup_{A \in \mathcal{A}'} \{\mu_A(x)\} = 1 \quad \forall x \in X.$$

Definition 5.2.5. Finite Intersection Property [Cha68]
Let U be a quasi fts and let $\mathcal{A} = \{A_\alpha | \alpha \in J\}$ be a family of fuzzy sets. Then, $\mathcal{A}$ has **finite intersection property** if and only if there is a finite subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that the intersection

$$\bigcap_{A \in \mathcal{A}'} A \neq \emptyset_X.$$

Theorem 5.2.6. [Cha68]
A quasi fts U is quasi F-compact if and only if each family of closed fuzzy sets that finite intersection property has a nonempty intersection.

Proof. Assume U is quasi F-compact and let $\mathcal{A}$ be a family of closed fuzzy sets in U. Suppose the intersection of all closed sets in $\mathcal{A}$ is empty. Namely,

$$\bigcap_{A \in \mathcal{A}} A = \emptyset_X.$$

By De Morgan's Law, we have

$$\bigcup_{A \in \mathcal{A}} A^c = \sqcup_X.$$

Since U is quasi F-compact and the open set A^c forms an open cover of U, then there exists a finite family $\mathcal{A}' \subseteq \mathcal{A}$ such that

$$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

By applying De Morgan's Law again, we have

$$\bigcap_{A \in \mathcal{A}'} A = \emptyset_X.$$

This contradicts the finite intersection property. Hence, the intersection of all closed sets in $\mathcal{A}$ must be nonempty.

Conversely, assume that every family of closed fuzzy sets that satisfies the finite intersection property has a nonempty intersection. Let $\mathcal{A}$ be an open cover of U. Then, let $\mathcal{B} = \mathcal{A}^c$ be the family of closed fuzzy sets corresponding to $\mathcal{A}$. Since $\mathcal{A}$ covers U, we must have

$$\bigcap_{A^c \in \mathcal{B}} A^c = \emptyset_X.$$

By definition 5.2.5, there exists a finite subfamily $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$\bigcap_{A^c \in \mathcal{B}'} A^c = \emptyset_X.$$

By De Morgan's Law, there exists a finite subfamily $\mathcal{A}' \subseteq \mathcal{A}$ corresponding to $\mathcal{B}'$ such that

$$\bigcup_{A \in \mathcal{A}'} A = \sqcup_X.$$

Hence, U is quasi F-compact. □

Theorem 5.2.7. [Cha68]
Let f be a F-continuous function mapping the quasi F-compact quasi fts $U_1 = (X, \delta)$ onto quasi fts $U_2 = (Y, \gamma)$. Then, U_2 is quasi F-compact.

Proof. Let $\mathcal{B}$ be an open cover of U_2. Then, we have

$$\bigcup_{B \in \mathcal{B}} \mu_{f^{-1}[B]}(x) = \sup_{B \in \mathcal{B}} \{\mu_{f^{-1}[B]}(x)\} = \sup_{B \in \mathcal{B}} \{\mu_B f(x)\} = 1 \quad \forall x \in X.$$

For all $B \in \mathcal{B}$, the family of all fuzzy sets of the form $f^{-1}[B]$ is an open cover of X which has a finite subcover. Since f is onto, for all $B \in \mathcal{B}$, there exists an open fuzzy set $A \in U_1$ such that $f[A] = B$ and we have $A = f^{-1}[B]$. Hence, the family of images of members of the subcover is a finite subfamily of $\mathcal{B}$ that covers U_2. Thus U_2 is quasi F-compact. $\square$

5.3 Compactness of Fuzzy Topological Spaces

In [Cha68], Chang gives a definition of compactness for quasi fts which formally follow the definition of compactness in general topology. Chang's definition also be used in [Won73] and [Gog73]. However, under this definition $(X, \mathscr{T})$ is compact does not implies $\big(X, \omega(\mathscr{T})\big)$ is compact.

Let us look at a counter-example.

Example 5.3.1. Consider $(X, \mathscr{T}) = I_r$ be the unit interval $X = I$ with the usual topology. A function y is linear if it has the form $y = mx + c$ for some $m,\ c \in \mathbb{R}$. Then for every $x \in X$ where $x \neq 0$ and $x \neq 1$, we define $\mu_x(y)$ by the following:

$$\mu_x(y) = \begin{cases} 1 & \text{, if } y = x, \\ 0 & \text{, if } y \in [0, \frac{x}{2}] \cup [\frac{x+1}{2}, 1], \\ \frac{2}{x}y - 1 & \text{, if } y \in [\frac{x}{2}, x], \\ -\frac{2}{1-x}y + 1 + \frac{2x}{1-x} & \text{, if } y \in [x, \frac{x+1}{2}]. \end{cases}$$

We also define for

$$x = 0,\ \mu_0(y) = -y + 1\ \forall y \in I,$$
$$x = 1,\ \mu_1(y) = y \qquad \forall y \in I.$$

Then for all $x \in I$, we have $\mu_x \in \omega(\mathscr{T})$ and the following property:

$$\sup_{x \in I}\{\mu_x(y)\} = 1\ \ \forall y \in I.$$

However, there is not subfamily of $\omega(\mathscr{T})$ has this property.

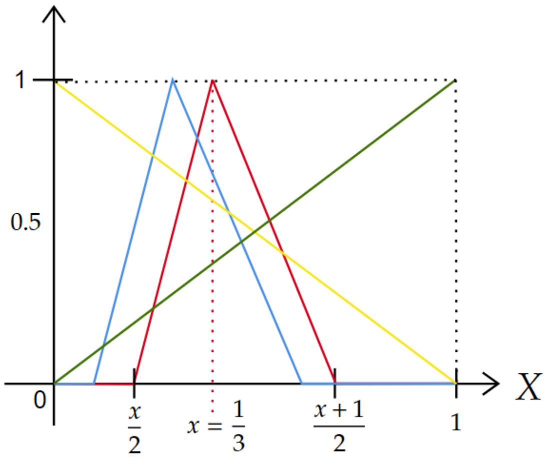

Figure 5.1: Counter-example under Chang's Definition.

Hence, Lowen has induced another form of compactness in [Low76].

Definition 5.3.2. Fuzzy Compact as A Fuzzy Set [Low76]
Let (X, δ) be a fts (or quasi fts). A fuzzy set $B = (X, \mu_B)$ is said to be a **fuzzy compact set**, or F-compact set in short, if for all family $\mathcal{A} \subset \delta$ such that

$$\sup_{A \in \mathcal{A}} \{\mu_A(x)\} \geq \mu_B(x) \quad \text{for all } x \in X,$$

and for all $\epsilon > 0$, there exists a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that

$$\max_{A \in \mathcal{A}'} \{\mu_A(x)\} = \sup_{A \in \mathcal{A}'} \{\mu_A(x)\} \geq \mu_B(x) - \epsilon \quad \text{for all } x \in X.$$

Definition 5.3.3. Fuzzy Compact as A Fuzzy Topological Space [Low76]
A fts (or quasi fts) (X, δ) is said to be **fuzzy compact** if each constant fuzzy set in (X, δ) is F-compact set.

Example 5.3.4. No fts can be quasi F-compact.
For a fts, $\sqcup_X$ has a covering $\mathcal{A} = \{\text{ constant } \alpha \,|\, \forall \alpha \in [0, 1)\}$. But there is no finite subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that $\sqcup_X = \bigcup_{A \in \mathcal{A}'} A$.

Definition 5.3.5. Weakly Fuzzy Compact [Low76]
A fts (or quasi-fts) (X, δ) is said to be **weakly *F*-compact** if $\sqcup_X$ is F-compact set.

Theorem 5.3.6. [Low76]
A fuzzy topological space $\big(X, \omega(\mathscr{T})\big)$ if F-compact if and only if $(X, \mathscr{T})$ is compact.

Proof. Recall that a fts is F-compact if every constant fuzzy set α in fts is a F-compact set. Assume $(X, \mathscr{T})$ is compact. Let $\beta \subset \omega(\mathscr{T})$ such that $\sup_{\mu\in\beta}\{\mu\} \geq \alpha > 0$ and let $\epsilon > 0$ such that $\alpha > \epsilon > 0$. For all $\mu \in \beta$, let $\mu^\epsilon = \mu + \epsilon$ and $[0, \alpha] = I_\alpha$. Then, $\forall \mu \in \beta$,

$$\mathscr{Q}(\mu^\epsilon) = \{(x, \alpha) \mid \mu^\epsilon(x) > \alpha\}$$

is an open set in $X \times I_\alpha$. Since $\bigcup_{\mu\in\beta} \mu^\epsilon = \sup_{\mu\in\beta}\{\mu^\epsilon\} = \sup_{\mu\in\beta}\{\mu\} + \epsilon \geq \alpha$, hence we have

$$X \times I_\alpha \subset \bigcup_{\mu\in\beta} \mathscr{Q}(\mu^\epsilon).$$

This shows the family $\{\mathscr{Q}(\mu^\epsilon) \mid \mu \in \beta\}$ forms an open cover of $X \times I_\alpha$. Moreover, $X \times I_\alpha$ is compact in general topological sense, then there exists a finite subfamily $\beta' \subset \beta$ such that

$$X \times I_\alpha \subset \bigcup_{\mu\in\mathcal{B}'} \mathscr{Q}(\mu^\epsilon).$$

Hence we have $\sup_{\mu\in\beta}\{\mu\} \geq \alpha - \epsilon$ which shows $\big(X, \omega(\mathscr{T})\big)$ is F-compact.

Conversely, assume $\big(X, \omega(\mathscr{T})\big)$ is F-compact. Suppose $\mathcal{A} \subset \mathscr{T}$ is an open cover of X. Then we have

$$X \subset \bigcup_{A\in\mathcal{A}} A \Leftrightarrow \sup_{A\in\mathcal{A}}\{\chi_A(x)\} = 1, \quad \forall x \in X$$

where χ_A is the characteristic function of A. Then, choose $\epsilon \in (0, 1)$. Notice that χ_A is a special case of fuzzy set. Since $\big(X, \omega(\mathscr{T})\big)$ is F-compact, then there exists $\mathcal{A}' \subset \mathcal{A}$ such that

$$\sup_{A\in\mathcal{A}'}\{\chi_A\} \geq 1 - \epsilon.$$

By definition of F-compact, $\mathcal{A}'$ is a finite subfamily of $\mathcal{A}$. Hence, $(X, \mathscr{T})$ is compact. $\square$

Proposition 5.3.7. [Low76]
If $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is F-continuous and μ is a F-compact set in (X, δ), then $f(\mu)$ is a F-compact set in (Y, γ).

Proof. Let $\beta \subset \gamma$ such that $\sup_{\nu\in\beta}\{\nu\} \geq f(\mu)$, then we have $\sup_{\nu\in\beta}\{f^{-1}(\nu)\} \geq \mu$. Since f is F-continuous, then $f^{-1}(\nu) \in \delta$. Notice that μ is a F-compact set, then for all $\epsilon > 0$ there exists a finite subfamily $\beta' \subset \beta$ such that

$$\sup_{\nu\in\beta'}\{f^{-1}(\nu)\} \geq \mu - \epsilon.$$

Then we have

$$\sup_{\nu\in\beta'}\{\nu\} \geq f(\mu - \epsilon) = f(\mu) - \epsilon$$

which shows $f(\mu)$ is a F-compact set in (Y, γ). □

Moreover, since the inverse of the constant fuzzy set is still a constant function, then we have the following proposition.

Proposition 5.3.8. [Low76]
If (X, δ) is F-compact and $\tilde{f} : (X, \delta) \to (Y, \gamma)$ is F-continuous, then (Y, γ) is F-compact.

Proof. Recall that a fts is F-compact if every constant fuzzy set is F-compact set. If (X, δ) is F-compact, then every constant fuzzy set in δ is a F-compact set. By definition 4.3.1, the inverse of a constant fuzzy set is still a constant fuzzy set. By proposition 5.3.7, for every constant fuzzy set β in γ, there is some F-compact constant fuzzy set α in δ such that $\beta = f(\alpha)$ is a F-compact set. Hence we have (Y, γ) is F-compact. □

Proposition 5.3.9. [Low76]
If (X, δ) is topologically generated F-compact, i.e. there exists a compact topology $\mathscr{T}$ such that $\delta = \omega(\mathscr{T})$, then every closed fuzzy set is a F-compact set.

Proof. Consider constant α such that $\alpha^c \in \omega(\mathscr{T})$ and $\beta \subset \omega(\mathscr{T})$ such that $\sup_{\mu\in\beta}\{\mu\} \geq \alpha$. Since $\alpha^c \in \omega(\mathscr{T})$, then we have

$$\mathscr{U}(\alpha) = \{(x, r) \,|\, \alpha(x) < r\} \text{ is an open set in } X \times I.$$

Notice that $X \times I = \mathscr{U}(\alpha) \cup \mathscr{U}(\alpha)^c$. Since $X \times I$ is compact and $\mathscr{U}(\alpha)^c$ is a closed subset of $X \times I$, thus $\mathscr{U}(\alpha)^c$ is compact. Choose $\epsilon > 0$ and let

$$\mu^\epsilon = \mu + \epsilon.$$

Notice that the value of μ^ϵ can be larger than 1. Then $\forall \mu \in \beta$, we define

$$\mathscr{Q}(\mu^\epsilon) = \{(x, r) \mid \mu^\epsilon(x) > r\} \subset X \times I$$

is an open set in $X \times I$. Then we have $\mathscr{U}(\alpha)^c \subset \bigcup_{\mu \in \beta} \mathscr{Q}(\mu^\epsilon)$. This shows the family $\{\mathscr{Q}(\mu^\epsilon) \mid \mu \in \beta\}$ is an open covering of $\mathscr{U}(\alpha)^c$. Since $\mathscr{U}(\alpha)^c$ is compact, then there exists finite subfamily $\beta' \subset \beta$ such that

$$\mathscr{U}(\alpha)^c \subset \bigcup_{\mu \in \beta'} \mathscr{Q}(\mu^\epsilon).$$

Since $\sup_{\mu \in \beta}\{\mu\} \geq \alpha$, then $\sup_{\mu \in \beta}\{\mu^\epsilon\} \geq \alpha$. This shows $\sup_{\mu \in \beta}\{\mu\} \geq \alpha - \epsilon$. Hence, every closed fuzzy set is a F-compact set. □

Remark 5.3.10. A F-compact (X, δ) is not necessarily that every closed fuzzy set is a F-compact set. Counter-example will be given later.

Recall Alexander Subbasis Theorem in Section 5.1, theorem 5.1.7, we have a corresponding theorem in fuzzy topological space. It will be used to study the products of F-compact space. Let us look at the definition of finite character.

Definition 5.3.11. Finite Character [SF96]
A family $\mathcal{F}$ of sets is of **finite character** if

1. For each $A \in \mathcal{F}$, every finite subset of A is belongs to $\mathcal{F}$.

2. If every finite subset of a given set A belongs to $\mathcal{F}$, then A belongs to $\mathcal{F}$.

Lemma 5.3.12. Tukey Lemma
Let S be a non-empty set of finite character. Then S has an element which is maximal with respect to the subset relation.

Proof. This lemma follows the axiom of choice, which is an equivalent of the Well-Ordering Principle. □

Theorem 5.3.13. [Low76]
(X, δ) is F-compact if and only if for any subbasis σ for δ, for any $\beta \subset \sigma$ and for any $\alpha > \epsilon > 0$ such that $\sup_{\mu \in \beta}\{\mu\} \geq \alpha$, there exists a finite subfamily $\beta' \subset \beta$ such that

$$\sup_{\mu \in \beta'}\{\mu\} \geq \alpha - \epsilon.$$

Proof. The proof of this theorem can be found in Lowen's paper, Section 4, Theorem 4.6 [Low76]. □

Theorem 5.3.14. [Low76]
(X, δ) is weakly F-compact if and only if for any subbasis σ for δ, for any $\beta \subset \sigma$ such that $\sup_{\mu \in \beta}\{\mu\} = 1$ and for all $\epsilon > 0$, there exists a finite subset $\beta' \subset \beta$ such that

$$\sup_{\mu \in \beta'}\{\mu\} \geq 1 - \epsilon.$$

Proof. The proof of this theorem is similar to the proof of Theorem 5.3.13 with the restriction $\alpha = 1$. □

Example 5.3.15. In remark 5.3.10, we have noticed that if μ is closed in a F-compact (X, δ), then μ need not to be a F-compact set. Let us look at an example: Let $X = I$ and δ be the fuzzy topology with the following subbasis

$$\{\text{constant } \alpha\} \cup \{\mu_n \,|\, n \in \mathbb{N}\} \cup \{\mu \cup \mu^c\}$$

where for all $n \in \mathbb{N}$,

$$\mu_n(x) = \begin{cases} \frac{1}{3}, & \text{for all } x \in [0, \frac{1}{2} - \frac{1}{n+1}] \cup [\frac{1}{2} + \frac{1}{n+1}, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(x) = \begin{cases} 0, & \text{if } x = \frac{1}{2}; \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

By theorem 5.3.13, for any constant a, there exists some a finite family β' of σ such that

$$\sup_{\alpha \in \beta'}\{\alpha\} \geq a - \epsilon.$$

Hence, (X, δ) is F-compact. Notice that $\sup_{n \in \mathbb{N}}\{\mu_n\} = \mu$ and μ is a closed fuzzy set since $\mu^c \in \delta$. However, for any $0 < \epsilon < \frac{1}{3}$, there is no finite subfamily of $\{\mu_n \,|\, n \in \mathbb{N}\}$ that covers $(\mu - \epsilon) \cup 0$.

Let us look at another example. By theorem 5.3.6, $\big(X, \omega(\mathscr{T})\big)$ is F-compact if and only if $(X, \mathscr{T})$ is compact. Obviously, if $\big(X, \iota(\delta)\big)$ is compact, then (X, δ) is F-compact. However, the converse is not true.

Example 5.3.16. Let $X = I$ and δ be the fuzzy topology with subbasis

$$\{\text{constant } \alpha\} \cup \{\nu \,|\nu(x) = x \text{ or } 0,\, \forall x \in X\} \cup \{\chi_0\}$$

where χ_0 is the Dirac function at 0:

$$\chi_0(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

By theorem $5.3.13$, it is obvious to see that (X, δ) is F-compact by taking finite subfamily $\{\alpha' \,|\, \alpha' = \alpha - \epsilon\}$ where $\{\alpha'\}$ only contains one elements. However, since $\{\nu\} \in \sigma$, $\iota(\delta)$ is discrete. Hence, $\iota(\delta)$ is not compact.

CHAPTER 6

CONCLUSION

6.1 Applications of Fuzzy Mathematics

One significant application of fuzzy topological spaces is fuzzy decision making system [BY20]. This decision-making-system is the collection of single or multicriteria techniques aiming at selecting the best alternative in case of imprecise, incomplete, and vague data.

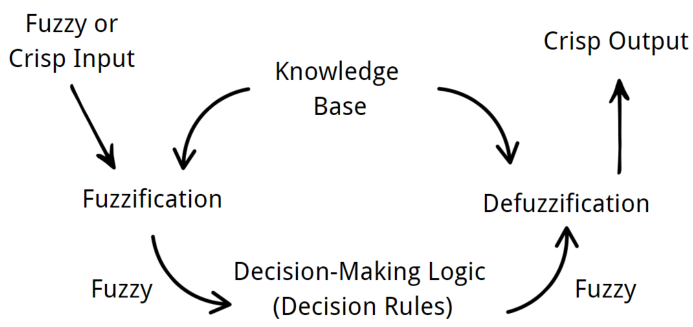

Figure 6.1: Basic Configuration of Fuzzy Decision-Making-System

Moreover, in control systems, fuzzy topological concepts are used to design fuzzy controllers that manage complex systems, like autonomous vehicles or industrial machinery, where exact parameters may be unavailable.

In image processing [Per15], fuzzy topology improves edge detection and noise reduction by considering gradual transitions between image regions instead of sharp boundaries. In artificial intelligence and machine learning, it supports inference, clustering and classification tasks under uncertainty, especially for datasets with fuzzy boundaries.

Additionally, fuzzy topological spaces find applications in economics, biological sciences, and social network analysis, where relationships and behaviors often exhibit uncertainty. Their ability to generalize traditional topology while incorporating fuzziness makes them a powerful tool for modeling real-world systems with inherent imprecision.

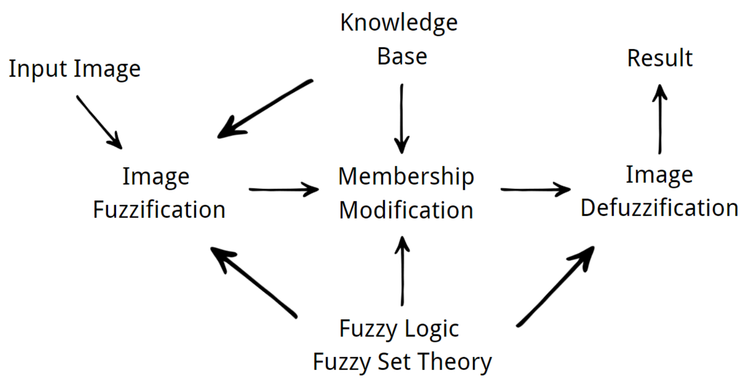

Figure 6.2: Basic Configuration of Fuzzy Image Processing

6.2 Potential Studies

The study of fuzzy topological spaces offers great potential for the development of theoretical and applied mathematics. It also provides valuable extensions to general topology. By incorporating fuzziness into topological structures, fuzzy topology provides a powerful framework for modeling uncertainty and imprecision in complex systems. This work highlights the fundamental properties of fuzzy topological spaces and their applications in various fields.

In modern world, AI plays an increasingly important role such as OpenAI's Chat GPT and Google's Gemini, so that the importance of fuzzy mathematics cannot be overemphasized. By generalizing general topological concepts to fuzzy environments, we can deepen our understanding of the continuity, compactness and convergence of spaces reflecting the real-world dimensions of ambiguity. Moreover, since computer computing systems, especially in the field of fuzzy control theory and decision making, need to operate under fuzzy conditions, fuzzy topology will definitely play an important role in the future.

In the fields of data science, machine learning and artificial intelligence, fuzzy topologies provide a natural way to deal with noisy, incomplete or imprecise data, leading to more robust analyses and decisions. Their application also extends to the fields of biological, social and economic modeling, where plays a crucial role in fuzzy system behavior.

Finally, the continued exploration of fuzzy topological spaces will lead to significant innovations in mathematical theory and practical applications, especially in areas where a more flexible and detailed description of reality is required. As this area of research continues to grow, it is expected to further our understanding of mathematical structures and the complex systems we have tried to model and control.

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